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LETTER TO THE EDITOR

Finite-difference equations in relativistic quantum mechanics*

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Abstract. Relativistic quantum mechanics suffers from structural problems which are traced back to the lack of a position operator \hat{x} , satisfying $[\hat{x}, \hat{p}] = i\hbar \hat{1}$ with the ordinary momentum operator \hat{p} , in the basic symmetry group—the Poincaré group. In this letter we provide a finite-dimensional extension of the Poincaré group containing only one more generator $\hat{\pi}$ (in $1+1$ dimensions), satisfying the commutation relation $[\hat{k}, \hat{\pi}] = i\hbar \hat{1}$ with the ordinary boost generator \hat{k} . The unitary irreducible representations are calculated and the carrier space proves to be the set of Shapiro's wavefunctions. The generalized equations of motion constitute a simple example of exactly solvable finite-difference set of equations associated with infinite-order polarization equations.

We begin with higher-order polarizations. One essential difference between geometric quantization [1–4] and a group approach to quantization (GAQ) [5, 6] is the possibility of introducing in the latter higher-order polarizations made out of elements of the (left) enveloping algebra. These higher-order polarizations are especially suitable for those (anomalous) cases in which the symplectic phase space is not polarizable, i.e. there is no maximal (half the dimension of the manifold) isotropic distribution of vector fields with respect to the symplectic form ω , or where there is no Lagrangian submanifold.

The GAQ is formulated on a group \tilde{G} which is a principal bundle with fibre $U(1)$ and the symplectic form is replaced by $d\Theta$, where Θ is the left 1-form dual to the vertical generator. The analogous, anomalous problem on a group, corresponds to the absence of a first-order full polarization, i.e. a maximal left subalgebra containing the kernel of Θ and excluding the $U(1)$ generator. It can be solved by adding operators in the left enveloping algebra to a non-full, first-order polarization, thus defining a higher-order polarization [6]. For instance the anomalous representations of the Virasoro group related to the non-Kählerian co-adjoint orbits have been successfully worked out in this way [7].

Higher-order polarizations have also proved to be useful for representing a physical system in a different (although equivalent) realization than those given by first-order, full polarizations. This is the case, for instance, for the configuration-space representation of the free particle and the harmonic oscillator (either relativistic or not) [8, 9] whose first-order polarizations lead to momentum-space and Bargmann–Fock representations, respectively.

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An interesting particular of the infinite-order polarization is the finite-difference character of the generalized equations of motion and physical operators, in some sense analogous to the differential realization of quantum group operators [10, 11]. It constitutes a system of infinite-order differential operators closing a 'weak' algebra: in general the commutator of two operators only closes on the reduced space of wavefunctions.

The general form of the solutions to the polarization equations, leading to the q -realization, can be written formally as

$$\Psi(t, p, q) \approx e^{-ip\hat{O}_p} e^{-it\hat{O}_t} \psi(q) \quad (1)$$

where t represents the variables with non-dynamical character like time, rotations, etc, q the generalized coordinates and p their conjugated momenta. $\psi(q)$ is an arbitrary function and constitutes the carrier space for the irreducible representation of the group. \hat{O}_t is the set of operators generating transformations in the variables t (energy, angular momentum, etc) and \hat{O}_p is a set of operators generating (no-local) transformations in p , all of them given in the q -realization. When the set of operators $\{\hat{O}_t, \hat{O}_p\}$ generates (under commutation) a subalgebra of the original Lie algebra the unitarity of the minimal realization in terms of only q and $\frac{\partial}{\partial q}$ is ensured, otherwise a further unitarization process is required, which consists of a non-canonical choice among algebraically equivalent higher-order polarizations, i.e. with the same commutation relations ('symmetrization' process). It must be stressed that the representation of the group on the complete wavefunction is nevertheless unitary. A general study of the integration of higher-order polarization will be published elsewhere.

In the particular case of relativistic quantum mechanics, the p -realization (momentum space) is naturally unitary, due to the fact that $\{\hat{O}_t, \hat{O}_x\}$ generates an algebra isomorphic to the one generated by $\{\hat{H}, \hat{p}\}$, which is a subalgebra of the Poincaré algebra. However, the x -realization (configuration space) requires further unitarization since $\{\hat{O}_t, \hat{O}_p\}$ generate an infinite-dimensional algebra isomorphic to the one generated by $\{\hat{H}, \hat{x}\}$, where \hat{x} is the position operator (note that $[\hat{x}, \hat{H}] = i\frac{\hat{p}}{\hat{H}} \approx i\frac{\hat{p}}{mc} - i\frac{\hat{p}^3}{2m^3c^3} + \dots$).

In this letter we elaborate on a finite enlargement of the $(1+1)$ D Poincaré group which contains a momentum operator $\hat{\pi}$ giving a canonical commutation relation with the boost operator \hat{k} . An infinite-order polarization on a central extension of this group provides a unitary representation of the Poincaré subgroup in a natural way, the support of which are the Shapiro's wavefunctions. It constitutes an example of an exactly tractable physical system associated with finite-difference (generalized) equations of motion and an alternative way out to the problem of the position operator.

We now turn to the momentum-space representation of the $(1+1)$ D Poincaré group. Let us review very briefly (see [8] and references therein for more details) the standard momentum-space representation of the pseudo-extended $(1+1)$ D Poincaré group on the basis of GAQ. Our starting point is the group law for the ordinary (non-extended) Poincaré group P . It is easily derived from its action on the $(1+1)$ D Minkowski spacetime parametrized by $\{a^\mu\} \equiv \{a^0, a^1 = a\}$: $a'^\mu = \Lambda^\mu_\nu(p^0, p)a^\nu + x^\mu$, where $\{x^\mu\}$ are the translations and Λ , the boosts, are parametrized by either p or

$$\chi \equiv \sinh^{-1} \frac{p}{mc} \equiv \sinh^{-1} \left(\gamma \frac{V}{c} \right) \equiv 2 \sinh^{-1} \alpha.$$

χ is the hyperpolar co-ordinate parametrizing the (upper sheet of the) hyperboloid $p^{02} - p^2 = m^2c^2$, often referred to as the *Lobachevsky space* (see [10] and references therein). In terms

of p

$$\Lambda = \left[\begin{array}{cc} \frac{p^0}{mc} & \frac{p}{mc} \\ \frac{p}{mc} & 1 + \frac{p^2}{mc(p^0 + mc)} \end{array} \right].$$

As a manifold, the group can be seen as the direct product of Minkowski spacetime and the mass hyperboloid.

The consecutive action of two Poincaré transformations leads to the composition law

$$\begin{aligned} x^{0''} &= x^{0'} + \frac{p^{0'}}{mc} x^0 + \frac{p'}{mc} x \\ x'' &= x' + \frac{p^{0'}}{mc} x + \frac{p'}{mc} x^0 \\ p'' &= \frac{p^0}{mc} p' + \frac{p^{0'}}{mc} p. \end{aligned} \tag{2}$$

The Poincaré group admits only trivial central extensions by $U(1)$, i.e. extensions of the form

$$\begin{aligned} g'' &= g' * g & g \in P \\ \zeta'' &= \zeta' \zeta e^{i\xi(g',g)} & \zeta \in U(1) \end{aligned}$$

where the cocycle ξ is a coboundary generated by a function η on P , $\xi(g', g) = \eta(g' * g) - \eta(g') - \eta(g)$. We choose $\eta(g) = mcx^0$, so that the $U(1)$ law to be added to (2) is

$$\zeta'' = \zeta' \zeta \exp [imc(x^{0''} - x^{0'} - x^0)]. \tag{3}$$

From (2) and (3) we immediately derive both left- and right-invariant vector fields:

$$\begin{aligned} \tilde{X}_{x^0}^L &= \frac{p^0}{mc} \frac{\partial}{\partial x^0} + \frac{p}{mc} \frac{\partial}{\partial x} + \frac{p^0(p^0 - mc)}{mc} \Xi \\ \tilde{X}_x^L &= \frac{p^0}{mc} \frac{\partial}{\partial x} + \frac{p}{mc} \frac{\partial}{\partial x^0} + \frac{p(p^0 - mc)}{mc} \Xi \\ \tilde{X}_p^L &= \frac{p^0}{mc} \frac{\partial}{\partial p} + \frac{p^0}{mc} x \Xi \\ \tilde{X}_\zeta^L &= i\zeta \frac{\partial}{\partial \zeta} \equiv \Xi \\ \tilde{X}_{x^0}^R &= \frac{\partial}{\partial x^0} \\ \tilde{X}_x^R &= \frac{\partial}{\partial x} + p \Xi \\ \tilde{X}_p^R &= \frac{p^0}{mc} \frac{\partial}{\partial p} + \frac{x^0}{mc} \frac{\partial}{\partial x} + \frac{x}{mc} \frac{\partial}{\partial x^0} + \left(\frac{p}{mc} x^0 + \frac{p^0}{mc} x - x \right) \Xi \\ \tilde{X}_\zeta^R &= i\zeta \frac{\partial}{\partial \zeta} \equiv \Xi. \end{aligned} \tag{4}$$

(5)

The pseudo-extended Poincaré algebra becomes

$$\begin{aligned} [\tilde{X}_{x^0}^R, \tilde{X}_x^R] &= 0 \\ [\tilde{X}_{x^0}^R, \tilde{X}_p^R] &= \frac{1}{mc} \tilde{X}_x^R \\ [\tilde{X}_x^R, \tilde{X}_p^R] &= \frac{1}{mc} \tilde{X}_{x^0}^R - \Xi. \end{aligned} \quad (6)$$

Note the appearance of the central generator in the third commutator above, making the extension by $U(1)$ not so trivial and justifying the name of pseudo-extension.

The pseudo-extended Poincaré group admits a first-order full polarization which is generated by $(\tilde{X}_{x^0}^L, \tilde{X}_x^L)$. The corresponding polarized $U(1)$ -functions ($\Xi\Psi = i\Psi$) are $\Psi = \exp[-i(p^0 - mc)x^0]\Phi(p)$ and the right generators act on them as quantum operators

$$\begin{aligned} \hat{p}^0 \Psi &\equiv i(\tilde{X}_{x^0}^R + mc\Xi)\Psi = p^0\Psi \quad \Rightarrow \quad \hat{p}^0 \Phi = p^0 \Phi \\ \hat{p} \Psi &\equiv -i\tilde{X}_x^R\Psi = p\Psi \quad \Rightarrow \quad \hat{p} \Phi = p\Phi \\ \hat{k} \Psi &\equiv i\tilde{X}_p^R\Psi = i\frac{p^0}{mc} e^{-i(p^0 - mc)x^0} \frac{\partial \Phi}{\partial p} \quad \Rightarrow \quad \hat{k} \Phi = i\frac{p^0}{mc} \frac{\partial \Phi}{\partial p}. \end{aligned} \quad (7)$$

In this identification of the quantum operator with right generators the rest mass energy has been added to the time generator to obtain the true energy operator \hat{p}^0 . This is a consequence of the fact that the pseudo-extension is nothing other than a redefinition of the $U(1)$ parameter. We must realize that the boost operator \hat{k} is not a true position operator, i.e. it is not $i\frac{\partial}{\partial p}$ and does not generate ordinary translations in the spectrum of the momentum operator \hat{p} .

We now turn to the S-Poincaré group in relativistic configuration space. The main problem we face in quantizing relativistic mechanics is the absence of a commutator like $[\hat{x}, \hat{p}] = i\hat{1}$ in the basic symmetry group, the Poincaré group, where we only find $[\hat{k}, \hat{p}] = i\hat{p}^0/mc$. As mentioned at the start of the letter, the position operator \hat{x} belongs to the infinite-order shell of the Poincaré algebra and does not close a finite-dimensional algebra with the rest of the generators. Furthermore, the operator \hat{p} does not generate translations on the spectrum of \hat{k} or, equivalently, the ordinary Minkowski variable x is not the spectrum of the operator \hat{k} .

The solution we propose in this letter is to keep \hat{k} as basic operator and look for a new momentum operator $\hat{\pi}$ such that

$$[\hat{k}, \hat{\pi}] = i\hat{1} \quad (8)$$

where $\hat{\pi}$ now closes a finite-dimensional enlarged Poincaré algebra: the S-Poincaré algebra. The operator $\hat{\pi}$ generates true translations on the spectrum of \hat{k} , κ . The spectrum of $\hat{\pi}$, $\pi \equiv mc\chi$, is related to p through

$$\pi \equiv mc \sinh^{-1} \frac{p}{mc} = mc \cosh^{-1} \frac{p^0}{mc}.$$

The group law we propose for the S-Poincaré group is given by

$$\begin{aligned}
 x^{0''} &= x^{0'} + \frac{p^{0'}}{mc} x^0 + \frac{p'}{mc} x \\
 x'' &= x' + \frac{p^{0'}}{mc} x + \frac{p'}{mc} x^0 \\
 p'' &= \frac{p^0}{mc} p' + \frac{p^{0'}}{mc} p \\
 \kappa'' &= \kappa' + \kappa \\
 \zeta'' &= \zeta' \zeta \exp[imc(x^{0''} - x^{0'} - x^0)] \exp\left[imc\kappa' \sinh^{-1} \frac{p}{mc}\right]
 \end{aligned} \tag{9}$$

The composition law for the new parameter κ is just additive and does not modify the first three lines (all four lines constituting the group law for the non-extended S-Poincaré group) since the associate operator $\hat{\pi}$ commutes with the whole non-extended Poincaré algebra. We can think of κ as parametrizing a Poincaré-invariant space. Furthermore, the extended commutator (8) requires a non-trivial cocycle in the composition law for ζ ; it takes the standard form $\exp(i\kappa'\pi)$. From this group law the left-invariant vector fields

$$\begin{aligned}
 \tilde{X}_{x^0}^L &= \frac{p^0}{mc} \frac{\partial}{\partial x^0} + \frac{p}{mc} \frac{\partial}{\partial x} + \frac{p^0(p^0 - mc)}{mc} \Xi \\
 \tilde{X}_x^L &= \frac{p^0}{mc} \frac{\partial}{\partial x} + \frac{p}{mc} \frac{\partial}{\partial x^0} + \frac{p(p^0 - mc)}{mc} \Xi \\
 \tilde{X}_p^L &= \frac{p^0}{mc} \frac{\partial}{\partial p} + \left(\kappa + \frac{p^0}{mc} x\right) \Xi \\
 \tilde{X}_\kappa^L &= \frac{\partial}{\partial \kappa} \\
 \tilde{X}_\zeta^L &= i\zeta \frac{\partial}{\partial \zeta} \equiv \Xi
 \end{aligned} \tag{10}$$

and the right-invariant ones

$$\begin{aligned}
 \tilde{X}_{x^0}^R &= \frac{\partial}{\partial x^0} \\
 \tilde{X}_x^R &= \frac{\partial}{\partial x} + p \Xi \\
 \tilde{X}_p^R &= \frac{p^0}{mc} \frac{\partial}{\partial p} + \frac{x^0}{mc} \frac{\partial}{\partial x} + \frac{x}{mc} \frac{\partial}{\partial x^0} + \left(\frac{p}{mc} x^0 + \frac{p^0}{mc} x - x\right) \Xi \\
 \tilde{X}_\kappa^R &= \frac{\partial}{\partial \kappa} + mc \sinh^{-1} \frac{p}{mc} \Xi \\
 \tilde{X}_\zeta^R &= i\zeta \frac{\partial}{\partial \zeta} \equiv \Xi
 \end{aligned} \tag{11}$$

are easily derived. The commutators between (say) right generators are

$$\begin{aligned}
 [\tilde{X}_{x^0}^R, \tilde{X}_x^R] &= 0 \\
 [\tilde{X}_{x^0}^R, \tilde{X}_p^R] &= \frac{1}{mc} \tilde{X}_x^R \\
 [\tilde{X}_{x^0}^R, \tilde{X}_\kappa^R] &= 0 \\
 [\tilde{X}_x^R, \tilde{X}_p^R] &= \frac{1}{mc} \tilde{X}_{x^0}^R - \Xi \\
 [\tilde{X}_x^R, \tilde{X}_\kappa^R] &= 0 \\
 [\tilde{X}_\kappa^R, \tilde{X}_p^R] &= -\Xi.
 \end{aligned} \tag{12}$$

The quantization 1-form, i.e. the $U(1)$ -left-invariant canonical 1-form

$$\begin{aligned}
 \Theta &= -\left(x + \frac{mc}{p^0} \kappa\right) dp - (p^0 - mc) dx^0 + \frac{d\zeta}{i\zeta} \\
 &\equiv -\left(x \cosh \frac{\pi}{mc} + \kappa\right) d\pi - mc \left(\cosh \frac{\pi}{mc} - 1\right) dx^0 + \frac{d\zeta}{i\zeta}
 \end{aligned} \tag{13}$$

provides the Noether invariants

$$\begin{aligned}
 i_{\tilde{X}_{x^0}^R} \Theta &= -(p^0 - mc) \\
 i_{\tilde{X}_p^R} \Theta &= p \\
 i_{\tilde{X}_x^R} \Theta &= -\frac{p^0}{mc} x + \frac{p}{mc} x^0 - \kappa \equiv -K \\
 i_{\tilde{X}_\kappa^R} \Theta &= mc \sinh^{-1} \frac{p}{mc} \equiv \pi.
 \end{aligned} \tag{14}$$

The integration measure, defined as the product of all left-invariant forms, is

$$\Omega = \frac{mc}{p^0} dx^0 \wedge dx \wedge dp \wedge d\kappa = dx^0 \wedge dx \wedge d\pi \wedge d\kappa. \tag{15}$$

The characteristic module of Θ (the Kernel of the Lie algebra cocycle) is $\mathcal{G}_\Theta = \langle \tilde{X}_{x^0}^L, \tilde{X}_x^L - \tilde{X}_\kappa^L \rangle$, and constitutes the generalized equation of motion. \mathcal{G}_Θ , or at least a proper subalgebra, must be included in any polarization.

We now present representations of the S-Poincaré group. There are three equivalent polarizations which come out naturally

$$\mathcal{P}_p^I = \left\langle \tilde{X}_{x^0}^L, \tilde{X}_x^L - \tilde{X}_\kappa^L, \tilde{X}_x^L + \tilde{X}_\kappa^L \right\rangle \approx \left\langle \tilde{X}_{x^0}^L, \tilde{X}_x^L, \tilde{X}_\kappa^L \right\rangle \tag{16}$$

$$\mathcal{P}_x^{\text{HO}} = \left\langle \tilde{X}_{x^0}^L + \left[\sqrt{m^2 c^2 - (\tilde{X}_x^L)^2} - mc \right] \Xi, \tilde{X}_x^L + mc \sinh^{-1} \left(\frac{i}{mc} \tilde{X}_x^L \right) \Xi, \tilde{X}_p^L \right\rangle \tag{17}$$

$$\mathcal{P}_\kappa^{\text{HO}} = \left\langle \tilde{X}_{x^0}^L + mc \left[\cosh \left(\frac{i}{mc} \tilde{X}_\kappa^L \right) - 1 \right] \Xi, \tilde{X}_x^L + mc \sinh \left(\frac{i}{mc} \tilde{X}_\kappa^L \right) \Xi, \tilde{X}_p^L \right\rangle. \tag{18}$$

They correspond to the realization of the unitary irreducible representation in momentum space, x -configuration space (ordinary configuration space) and κ -configuration space, respectively.

We now present the realization of the \mathcal{P}_p^I polarization in momentum space. The polarization equations (once ζ has been factorized out everywhere) give us the irreducible wavefunctions

$$\left. \begin{aligned} \tilde{X}_{x^0}^L \Psi &= 0 \\ \tilde{X}_x^L \Psi &= 0 \\ \tilde{X}_\kappa^L \Psi &= 0 \end{aligned} \right\} \Rightarrow \Psi = \exp[-i(p^0 - mc)x^0] \Phi(p) \quad (19)$$

on which the right generators act as quantum operators

$$\left. \begin{aligned} i\tilde{X}_{x^0}^R \Psi &= (p^0 - mc)\Psi \\ -i\tilde{X}_x^R \Psi &= p\Psi \\ i\tilde{X}_p^R \Psi &= i\frac{p^0}{mc} \exp[-i(p^0 - mc)x^0] \frac{\partial \Phi}{\partial p} \\ -i\tilde{X}_\kappa^R \Psi &= mc \sinh^{-1} \frac{p}{mc} \Psi \end{aligned} \right\} \quad (20)$$

$$\Rightarrow \left\{ \begin{aligned} \hat{p}^0 \Phi &= p^0 \Phi \equiv mc \cosh \frac{\pi}{mc} \Phi \\ \hat{p} \Phi &= p \Phi \equiv mc \sinh \frac{\pi}{mc} \Phi \\ \hat{k} \Phi &= i\frac{p^0}{mc} \frac{\partial \Phi}{\partial p} \equiv i\frac{\partial \Phi}{\partial \pi} \\ \hat{\pi} \Phi &= mc \sinh^{-1} \frac{p}{mc} \Phi \equiv \pi \Phi. \end{aligned} \right. \quad (21)$$

Note that the boost operator \hat{k} can now be written as $i\frac{\partial}{\partial \pi}$, i.e. it generates translations in the spectrum of $\hat{\pi}$, turning $\hat{\pi}$ into a 'good' momentum operator.

We now present the realization of the $\mathcal{P}_x^{\text{HO}}$ polarization in x -configuration space. We follow the same steps as before. The polarization equations give the irreducible wavefunctions

$$\tilde{X}_p^L \Psi = 0 \Rightarrow \Psi = \exp[-ixp] \exp\left[-imc\kappa \sinh^{-1} \frac{p}{mc}\right] \Phi(x^0, x, \kappa) \quad (22)$$

$$\left\{ \tilde{X}_\kappa^L + mc \sinh^{-1} \left(\frac{i}{mc} \tilde{X}_x^L \right) \Xi \right\} \Psi = 0 \Rightarrow -i\frac{\partial \Phi}{\partial \kappa} = mc \sinh^{-1} \left(\frac{-i}{mc} \frac{\partial}{\partial x} \right) \Phi$$

$$\Rightarrow \Phi(x^0, x, \kappa) = \exp\left[imc\kappa \sinh^{-1} \left(\frac{-i}{mc} \frac{\partial}{\partial x} \right)\right] \phi(x^0, x) \quad (23)$$

$$\left\{ \tilde{X}_{x^0}^L + \left[\sqrt{m^2 c^2 - (\tilde{X}_x^L)^2} - mc \right] \Xi \right\} \Psi = 0 \Rightarrow i\frac{\partial \phi}{\partial x^0} = \left[\sqrt{m^2 c^2 - \frac{\partial^2}{\partial x^2}} - mc \right] \phi$$

$$\Rightarrow \phi(x^0, x) = \exp\left\{-ix^0 \left[\sqrt{m^2 c^2 - \frac{\partial^2}{\partial x^2}} - mc \right]\right\} \varphi(x) \quad (24)$$

$$\Psi = \exp[-ixp] \exp\left[-imc\kappa \sinh^{-1} \frac{p}{mc}\right] \exp\left[imc\kappa \sinh^{-1} \left(\frac{-i}{mc} \frac{\partial}{\partial x}\right)\right] \\ \times \exp\left\{-imcx^0 \left[\sqrt{m^2c^2 - \frac{\partial^2}{\partial x^2}} - mc\right]\right\} \varphi(x). \quad (25)$$

The resulting quantum operators (restricted to $\varphi(x)$) are

$$\hat{p}^0 \varphi = \sqrt{m^2c^2 - \frac{\partial^2}{\partial x^2}} \varphi \\ \hat{p} \varphi = -i \frac{\partial}{\partial x} \varphi \\ \hat{k} \varphi = \left[\frac{x}{mc} \sqrt{m^2c^2 - \frac{\partial^2}{\partial x^2}} + \kappa + i \frac{x^0}{mc} \frac{\partial}{\partial x}\right] \varphi \\ \hat{\pi} \varphi = mc \sinh^{-1} \left(\frac{-i}{mc} \frac{\partial}{\partial x}\right) \varphi. \quad (26)$$

This realization (restricted to x and $\frac{\partial}{\partial x}$) is not unitary, even though the representation is unitary on the complete wavefunctions, because the \hat{k} operator is not Hermitian. This problem is solved by using a proper (higher-order) polarization on the Poincaré group [8]. The $\mathcal{P}_\kappa^{\text{HO}}$ polarization on the S-Poincaré group provides an alternative solution.

We now present the realization of the $\mathcal{P}_\kappa^{\text{HO}}$ polarization in κ -configuration space and the Shapiro wavefunctions. The polarization equations again give the irreducible wavefunctions

$$\tilde{X}_p^L \Psi = 0 \quad \Rightarrow \quad \Psi = \exp[-ixp] \exp\left[-imc\kappa \sinh^{-1} \frac{p}{mc}\right] \Phi(x^0, x, \kappa) \quad (27)$$

$$\left\{\tilde{X}_{x^0}^L + mc \left[\cosh\left(\frac{i}{mc} \tilde{X}_\kappa^L\right) - 1\right] \Xi\right\} \Psi = 0 \quad \Rightarrow \quad i \frac{\partial \Phi}{\partial x^0} = mc \left[\cosh\left(\frac{-i}{mc} \frac{\partial}{\partial \kappa}\right) - 1\right] \Phi \\ \Rightarrow \quad \Phi(x^0, x, \kappa) = \exp\left\{-imcx^0 \left[\cosh\left(\frac{-i}{mc} \frac{\partial}{\partial \kappa}\right) - 1\right]\right\} \phi(x, \kappa) \quad (28)$$

$$\left\{\tilde{X}_x^L + mc \sinh\left(\frac{i}{mc} \tilde{X}_\kappa^L\right) \Xi\right\} \Psi = 0 \quad \Rightarrow \quad -i \frac{\partial \phi}{\partial x} = mc \sinh\left(\frac{-i}{mc} \frac{\partial}{\partial \kappa}\right) \phi \\ \Rightarrow \quad \phi(x, \kappa) = \exp\left[imcx \sinh\left(\frac{-i}{mc} \frac{\partial}{\partial \kappa}\right)\right] \varphi(\kappa) \quad (29)$$

$$\Psi = \exp[-ixp] \exp\left[-imc\kappa \sinh^{-1} \frac{p}{mc}\right] \exp\left\{-imcx^0 \left[\cosh\left(\frac{-i}{mc} \frac{\partial}{\partial \kappa}\right) - 1\right]\right\} \\ \times \exp\left[imcx \sinh\left(\frac{-i}{mc} \frac{\partial}{\partial \kappa}\right)\right] \varphi(\kappa). \quad (30)$$

The quantum operators are

$$\begin{aligned}
 \hat{p}^0 \varphi &= mc \cosh \left(\frac{-i}{mc} \frac{\partial}{\partial \kappa} \right) \varphi \\
 \hat{p} \varphi &= mc \sinh \left(\frac{-i}{mc} \frac{\partial}{\partial \kappa} \right) \varphi \\
 \hat{k} \varphi &= \kappa \varphi - x^0 \sinh \left(\frac{-i}{mc} \frac{\partial}{\partial \kappa} \right) \varphi + x \cosh \left(\frac{-i}{mc} \frac{\partial}{\partial \kappa} \right) \varphi \\
 \hat{\pi} \varphi &= -i \frac{\partial}{\partial \kappa} \varphi.
 \end{aligned} \tag{31}$$

As can be seen, this representation of the S-Poincaré group, or more precisely the one which appears after dropping the x^0 and x 'evolution' from the \hat{k} operator, is unitary and contains a unitary and irreducible representation of the Poincaré subgroup.

An analogous comment to that made under (21) also applies here; put the other way round the new parameter κ allows \hat{k} to be written as a multiplicative operator.

The Shapiro wavefunctions are simply the eigenfunctions of the momentum operator \hat{p} in κ -configuration space, namely

$$\begin{aligned}
 \hat{p} \varphi_p(\kappa) = p \varphi_p(\kappa) &\Rightarrow mc \sinh \left(\frac{-i}{mc} \frac{\partial}{\partial \kappa} \right) \varphi_p = p \varphi_p \\
 \Rightarrow \varphi_p(\kappa) &= \exp \left[imc \kappa \sinh^{-1} \frac{p}{mc} \right] \equiv \exp[i\kappa \pi] = \left(\frac{p^0 - p}{mc} \right)^{-imc\kappa}.
 \end{aligned} \tag{32}$$

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